The finite subgroups of SU(3)

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Workshop on flavor symmetries, Dortmund, July 3rd, 2012

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FWF Der Wissenschaftsfonds.

FWF Project P 24161-N16

Finite groups in particle physics

Particle physics offers a wide range of applications for the theory of finite groups.

$\rightarrow$ In particular the finite subgroups of SU(3) have been intensively studied in the past.

- Model building in hadron physics.
- Computational tools in lattice QCD.
- Flavour physics: quark and lepton sector, scalar sector.

Selection of contributions

1916 Miller, Blichfeldt, Dickson: Theory and applications of finite groups: Classification of the finite subgroups of SU(3) in terms of their generators.

1964 Fairbairn, Fulton, Klink: Analyzed a large set of finite subgroups of SU(3) for their usage as symmetries in particle physics. $\Delta(3n^2)$, $\Delta(6n^2)$ mentioned.

1981 Bovier, Lüling, Wyler: T_n, $\Delta(3n^2)$, $\Delta(6n^2)$.

2007 Luhn, Nasri, Ramond: $\Delta(3n^2)$, $I \cong A_5$, $\widetilde{I}$, $\Sigma(168) \cong \mathrm{PSL}(2,7)$.

2008 Escobar, Luhn: $\Delta(6n^2)$.

2009 POL: Groups of types (C) and (D).
Zwicky, Fischbacher: Groups of type (D).

2010 POL: Finite subgroups of U(3) of order smaller than 512.
Ishimori, Kobayashi, Ohki, Okada, Shimizu, Tanimoto: Non-Abelian discrete symmetries in particle physics.
Parattu, Wingerter: All finite groups of order smaller 100.

2011 Grimus, POL: Structure of groups of types (C) and (D).
Luhn; Merle, Zwicky: Breaking of SU(3) to its finite subgroups.

The finite subgroups of SU(3)

H.F. Blichfeldt (1916)[1]:

Classification of the finite subgroups of SU(3) into five types:

(A) Abelian groups.

(B) Finite subgroups of SU(3) with faithful 2-dimensional representations.

(C) The groups $C(n, a, b)$ generated by the matrices

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad F(n, a, b) = \mathrm{diag}(\eta^a, \eta^b, \eta^{-a-b}),$$

where $\eta = \exp(2\pi i/n)$.

[1] G.A. Miller, H.F. Blichfeldt and L.E. Dickson: Theory and applications of finite groups, New York (1916)

(D) The groups $D(n,a,b;d,r,s)$ generated by E, $F(n,a,b)$ and

$$\widetilde{G}(d,r,s) = \begin{pmatrix} \delta^r & 0 & 0 \\ 0 & 0 & \delta^s \\ 0 & -\delta^{-r-s} & 0 \end{pmatrix},$$

where $\delta = \exp(2\pi i/d)$.

(E) Six exceptional finite subgroups of SU(3):

- $\Sigma(60) \cong A_5$, $\Sigma(168) \cong \mathrm{PSL}(2,7)$
- $\Sigma(36\times 3)$, $\Sigma(72\times 3)$, $\Sigma(216\times 3)$ and $\Sigma(360\times 3)$,

as well as the direct products $\Sigma(60)\times\mathbb{Z}_3$ and $\Sigma(168)\times\mathbb{Z}_3$.

(A) Abelian groups

Simple (but powerful) theorem:

Abelian finite subgroups of SU(3)

Every finite Abelian subgroup $\mathcal{A}$ of SU(3) is isomorphic to

$$\mathbb{Z}_m \times \mathbb{Z}_n,$$

where

$$m = \max_{a \in \mathcal{A}} \mathrm{ord}(a)$$

and n is a divisor of m.

$\Rightarrow$ Possible structures of Abelian finite subgroups of SU(3) are strongly restricted!

Examples:

- Rotations about one axis (cyclic groups $\mathbb{Z}_m$)
- Klein's four group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(B) Groups with two-dimensional faithful representations

For every finite subgroup of SU(2) there is an isomorphic finite subgroup of SU(3).

$$A \in \mathrm{SU}(2) \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \in \mathrm{SU}(3)$$

However, this is even true for the finite subgroups of U(2).

$$A \in \mathrm{U}(2) \Rightarrow \begin{pmatrix} \det A^* & 0 \\ 0 & A \end{pmatrix} \in \mathrm{SU}(3)$$

Examples:

- Dihedral groups D_n (finite subgroups of SO(3)).
- Double covers of the finite 3-dimensional rotation groups ($\widetilde{T}$, $\widetilde{O}$, $\widetilde{I}$, $\widetilde{D_n}$).

The groups of type (C)

Generated by

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad F(n,a,b) = \mathrm{diag}(\eta^a, \eta^b, \eta^{-a-b}),$$

where $\eta = \exp(2\pi i/n)$.

Structure: $F(n,a,b)$ diagonal $\Rightarrow EF(n,a,b)E^{-1}$ also diagonal.

$\Rightarrow$ Subgroup $N(n,a,b)$ of diagonal matrices is a normal subgroup.

$$\Rightarrow C(n,a,b) \cong N(n,a,b) \rtimes \mathbb{Z}_3.$$

We also know that $N(n,a,b)$ is an Abelian finite subgroup of SU(3), thus

$$C(n,a,b) \cong (\mathbb{Z}_m \times \mathbb{Z}_p) \rtimes \mathbb{Z}_3.$$

The groups of type (C)

$$C(n,a,b) \cong (\mathbb{Z}_m \times \mathbb{Z}_p) \rtimes \mathbb{Z}_3.$$

Special cases:

- $p = 1 \Rightarrow$ Groups of the type[2] $T_m \cong \mathbb{Z}_m \rtimes \mathbb{Z}_3$.
- $p = m \Rightarrow$ Groups of the type $(\mathbb{Z}_m \times \mathbb{Z}_m) \rtimes \mathbb{Z}_3 \cong \Delta(3m^2)$.

Examples:

- Well-known groups such as $A_4 \cong T \cong \Delta(12),\ \Delta(27),\ T_7,\ T_{13}$.
- Smallest group of type (C) which is neither of the form T_n nor of the form $\Delta(3n^2)$:

$$C(9,1,1) \cong (\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3.$$

[2] m must be a product of powers of primes of the form $6k+1$.

The groups of type (D)

The group $D(n, a, b; d, r, s)$ is generated by the generators of $C(n, a, b)$ and

$$\widetilde{G}(d, r, s) = \begin{pmatrix} \delta^r & 0 & 0 \\ 0 & 0 & \delta^s \\ 0 & -\delta^{-r-s} & 0 \end{pmatrix},$$

where $\delta = \exp(2\pi i/d)$.

W. Grimus, POL (2011)[3]: By means of a unitary transformation one obtains a different set of generators:

- Three diagonal matrices,
- and the two S_3-generators

$$E = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

[3] W. Grimus, POL: *Finite flavour groups of fermions*, *J. Phys. A* 45 (2012) 233001; [arXiv:1110.6376].

The groups of type (D)

$$\Rightarrow D(n,a,b;d,r,s) \cong N(n,a,b;d,r,s) \rtimes S_3$$
$$\Rightarrow D(n,a,b;d,r,s) \cong (\mathbb{Z}_m \times \mathbb{Z}_{m'}) \rtimes S_3.$$

Special cases:

- $m = m' \Rightarrow$ Groups of the type $(\mathbb{Z}_m \times \mathbb{Z}_m) \rtimes S_3 \cong \Delta(6m^2)$.

Examples:

- Well-known groups such as $S_4 \cong \Delta(24)$, $\Delta(54)$.
- Smallest group of type (D) which is neither a direct product nor of the form $\Delta(6n^2)$:

$$D(9,1,1;2,1,1) \cong (\mathbb{Z}_9 \times \mathbb{Z}_3) \rtimes S_3.$$

$$D(n,a,b;d,r,s) \cong N(n,a,b;d,r,s) \rtimes S_3,$$

- N is the normal subgroup of all diagonal matrices in the group.
- S_3 is generated by the matrices E and B.
- $\Rightarrow$ Every element of the group can be written as

$$FB^j E^k \qquad (F \in N;\ \ j = 0,1;\ \ k = 0,1,2)$$

$\rightarrow$ Allows to determine the dimensions of the irreps of the group[4].

Consider an irrep $\mathcal{D}$ of the group.

$$\mathcal{D}: \quad N \mapsto \bar{N}, \quad B \mapsto \bar{B}, \quad E \mapsto \bar{E}.$$

$\bar{N}$ is Abelian $\Rightarrow$ There is at least one simultaneous eigenvector x of all $\bar{F} \in \bar{N}$.

[4] W. Grimus, POL: *Finite flavour groups of fermions*, *J. Phys. A* 45 (2012) 233001; [arXiv:1110.6376].

Dimensions of the irreps of the groups of types (C) and (D)

$\bar{N}$ is Abelian $\Rightarrow$ There is at least one simultaneous eigenvector x of all $\bar{F} \in \bar{N}$.

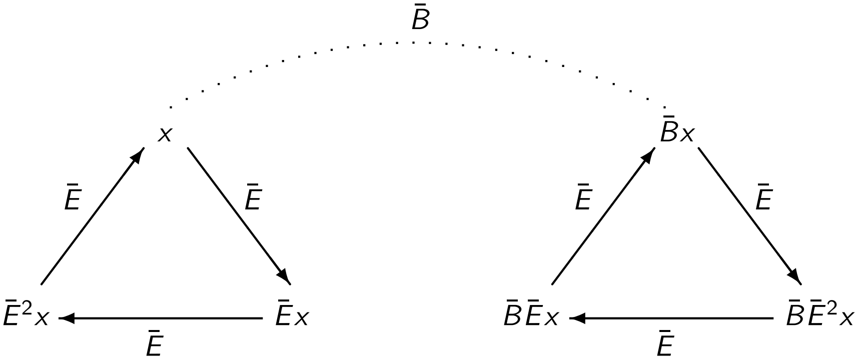
$\Rightarrow$ Also $\bar{E}x,\ \bar{E}^2x,\ \bar{B}x, \bar{B}\bar{E}x$ and $\bar{B}\bar{E}^2x$ simultaneous eigenvectors of all $\bar{F} \in \bar{N}$.

$\Rightarrow \{x,\ \bar{E}x,\ \bar{E}^2x,\ \bar{B}x, \bar{B}\bar{E}x, \bar{B}\bar{E}^2x\}$ closed under the action of the group.

$\mathcal{D}$ irreducible $\Rightarrow V_{\mathcal{D}} = \mathrm{span}\{x,\ \bar{E}x,\ \bar{E}^2x,\ \bar{B}x, \bar{B}\bar{E}x,\ \bar{B}\bar{E}^2x\}$

- $\Rightarrow$ Dimension of an irrep of a group of type (D) is at most 6.
- $\Rightarrow$ Dimension of an irrep of a group of type (C) is at most 3.

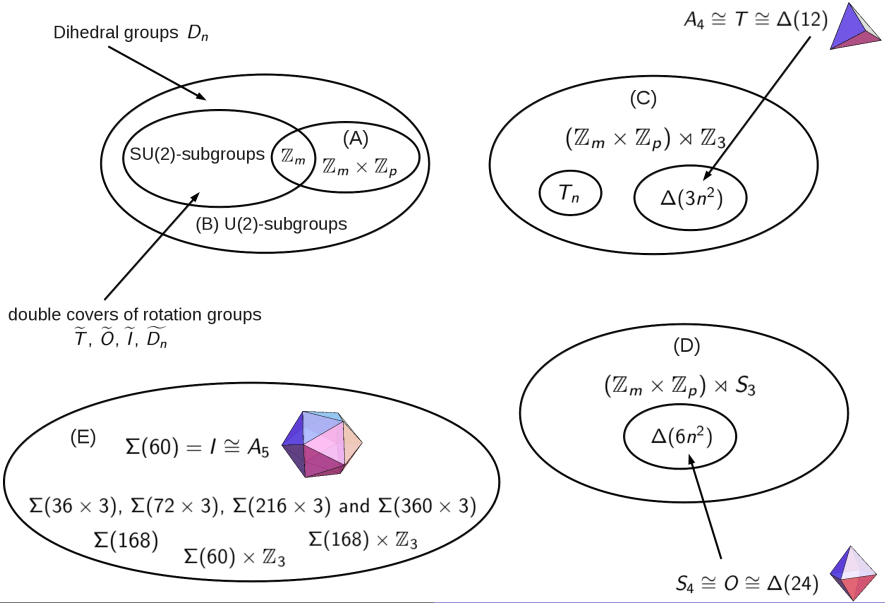
Dimensions of the irreps of the groups of types (C) and (D)



$\Rightarrow$ Dimension of an irrep of a group of type (D) is 1,2,3 or 6.

$\Rightarrow$ Dimension of an irrep of a group of type (C) is 1 or 3.

Summary: The finite subgroups of SU(3)



Thank you for your attention!