Group Theory in Theoretical Physics

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Outline of the talk

A. Where do we use group theory in theoretical physics?

B. Computational tools we use

C. Example from research

A. Where do we use group theory in theoretical physics?

Where do we use group theory in theoretical physics?

Symmetries of nature:

Language of physics is mathematics: Most equations that describe physical phenomena are partial differential equations (PDEs).

$\rightarrow$ Often describable by a so-called Lagrangian $L(x_i, \dot{x}_i)$:

$$\text{PDE (Euler-Lagrange equation)} \quad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0$$

Symmetries of nature $\leftrightarrow$ Symmetries of the Lagrangian L

$\leftrightarrow$ Discrete and continuous **symmetry groups**.

Example: Noether's theorem

Most fascinating result:

Theorem (Emmy Noether, 1918)

If the Lagrangian of a system is invariant under a continuous symmetry, then there exists a corresponding conservation law.

Example: Consider a system of two particles; Force between particles depends on the distance $r = |\vec{x}_1 - \vec{x}_2|$ only: described by potential $\phi(r)$.

$$\text{Newton's law:} \quad m_1 \ddot{\vec{x}}_1 = -\vec{\nabla}_{\vec{x}_1} \phi(r), \quad m_2 \ddot{\vec{x}}_2 = -\vec{\nabla}_{\vec{x}_2} \phi(r)$$

Corresponding Lagrangian:

$$L(\vec{x}_1, \dot{\vec{x}}_1, \vec{x}_2, \dot{\vec{x}}_2) = \frac{1}{2} m_1 \dot{\vec{x}}_1^2 + \frac{1}{2} m_2 \dot{\vec{x}}_2^2 - \phi(r)$$

Example: Noether's theorem

Now, symmetry: Move system of the two particles to another place in space $\Rightarrow$

$$\vec{x}_1 \to \vec{x}_1 + \delta\vec{x}, \quad \vec{x}_2 \to \vec{x}_2 + \delta\vec{x}, \quad \dot{\vec{x}}_1 \to \dot{\vec{x}}_1, \quad \dot{\vec{x}}_2 \to \dot{\vec{x}}_2, \quad r \to r.$$

$\Rightarrow$ Lagrangian unchanged $\Rightarrow$ Physics unchanged!

But this implies

$$L(\vec{x}_1 + \delta\vec{x}, \dot{\vec{x}}_1, \vec{x}_2 + \delta\vec{x}, \dot{\vec{x}}_2) = L(\vec{x}_1, \dot{\vec{x}}_1, \vec{x}_2, \dot{\vec{x}}_2)$$

and thus, if $\delta\vec{x}$ infinitesimal:

$$\sum_{i=1}^{3} \left(\frac{\partial L}{\partial x_{1i}} + \frac{\partial L}{\partial x_{2i}} \right) \delta x_i = 0$$

Valid for arbitrary δx_i.

$$\Rightarrow \left(\frac{\partial L}{\partial x_{1i}} + \frac{\partial L}{\partial x_{2i}} \right) = 0$$

Example: Noether's theorem

$$\left(\frac{\partial L}{\partial x_{1i}} + \frac{\partial L}{\partial x_{2i}}\right) = 0$$

Using Euler-Lagrange equation and inserting L this can be rewritten as

$$\frac{d}{dt}\underbrace{\left(m_1\dot{\vec{x}}_1 + m_2\dot{\vec{x}}_2\right)}_{\vec{P}} = 0.$$

Total momentum $\vec{P}$ of the system is conserved!

Invariance of L under translations (continuous group with 3 generators)

$\downarrow$

Momentum conservation (three components of $\vec{P}$).

Other examples: Rotational invariance $\rightarrow$ angular momentum conservation, time translation invariance $\rightarrow$ energy conservation.

Where do we use group theory in theoretical physics?

Examples for fields of theoretical physics where group theory is applied:

- High-energy (particle) physics
- Solid state physics: Lattices, crystal structures
- Different areas of mathematical physics

$\rightarrow$ Physicists look for symmetries inherent of problems:

$\rightarrow$ Allows to understand structure and classification of different solutions.

Also, nature shows many exact and approximate symmetries.

Most important example: Gauge symmetries: Exact symmetries of the Standard Model of particle physics.

E.g.: Electromagnetic gauge symmetry: Reason why electric charge is conserved and electromagnetic interaction has $1/r^2$-behaviour.

B. Tools we use

Theoretical tools

Mostly relevant for physics: Matrix representations of groups

$\rightarrow$ **Group representation theory**

Use all standard techniques like

- character tables,
- conjugacy classes,
- normal subgroups,
- factor groups,
- irreducible representations (irreps),
- group products (direct, semidirect)
- invariants
- ...

Computational tools

Talk about discrete groups now. Lie groups are a different story ...

For small groups character tables, irreps, *etc.* can be computed by hand.
For larger groups: Use computer algebra systems.

GAP: **G**roups, **A**lgorithms and **P**rogramming

www.gap-system.org

$\rightarrow$ Computer algebra system with scope of *discrete mathematics*:

finite groups, algebras, rings, fields, modules, number theory,...

GAP includes a programming language (interpreted; no compiler) with usual features: for, while, if, list manipulation, reading and writing files, ...

Typical use of GAP for group analysis

Group usually given as matrix group. Example:

```
A:=[[0, 1, 0],
    [0, 0, 1],
    [1, 0, 0]];

B:=[[1, 0, 0],
    [0,-1, 0],
    [0, 0,-1]];

g:=Group(A, B);
```

Finite groups

So, we have defined a group g. What can we do with it?

- `Order(g);`
- `StructureDescription(g);`
- `ConjugacyClasses(g);`
- `ct:=CharacterTable(g); Display(ct);`
- `IrreducibleRepresentations(g);`
- compute normal subgroups of g,
- compute all subgroups of g,
- express group elements in terms of generators (extremely useful),
- ...

The SmallGroups library

Another tool used frequently is the SmallGroups library.

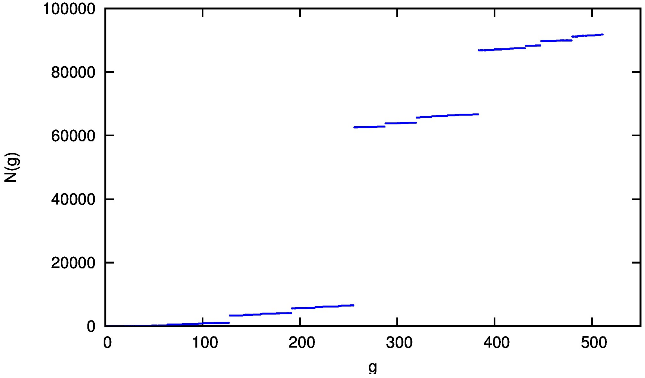
$\rightarrow$ Library of all finite groups up to order 2000 (except 1024).
$\rightarrow$ Included in GAP.

Can be used to find finite groups of small order (< 2000) with special properties.

The SmallGroups library

How many small finite groups are there?

$$N(g) \ldots \text{number of non-Abelian groups of order} \leq g.$$



Numbers soon get large. Particularly many groups of orders $2^m 3^n$, *e.g.* $\approx 5 \times 10^{10}$ groups of order $2^{10} = 1024$.

The SmallGroups library - Group scans

With additional restrictions like 3-dim. irreps, ...

$\rightarrow$ *Group scans* up to order $\sim 10^3$ become feasible.
$\rightarrow$ Frequently used in the literature.

Procedure:

- Choose desired properties of groups,
- Use GAP and the SmallGroups library to extract all groups up to a given order (< 2000) fulfilling the conditions.

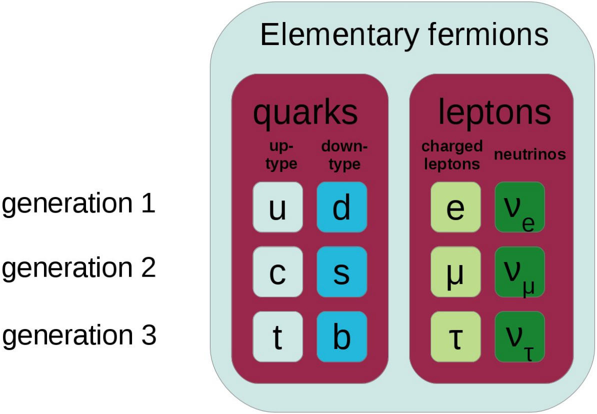
Powerful tool, but only groups up to some maximal order can be investigated.

C. Example from research

Flavour symmetries

→ What is flavour?

All known fermions exist in three different generations (flavours).



Reason: unknown.

Flavour symmetries

Particle physics based on quantum field theory: Basic objects are (operator-valued) fields living on 4-dimensional spacetime.

Theory described by a Lagrangian

$$\mathcal{L} = \mathcal{L}(e,\, \mu,\, \tau,\, \nu_e,\, \nu_\mu,\, \nu_\tau,\, \ldots)$$

$e(x)$, $\mu(x)$, $\tau(x)$, $\nu_e(x)$, … are *quantum fields*.

Flavour symmetries: Lagrangian invariant under the *flavour symmetry* transformations

$$\begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix} \to R_\ell(g) \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}, \quad \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} \to R_\nu(g) \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}, \quad \ldots$$

$G \ni g \mapsto R_\ell(g)$, $G \ni g \mapsto R_\nu(g), \ldots$ are three-dimensional unitary representations of a *flavour symmetry group* G.

Flavour symmetries

Flavour symmetries restrict the form of $\mathcal{L}$. To construct $\mathcal{L}$: Choose flavour symmetry group, choose representations, compute invariants. [$\rightarrow$ can use GAP as a tool]

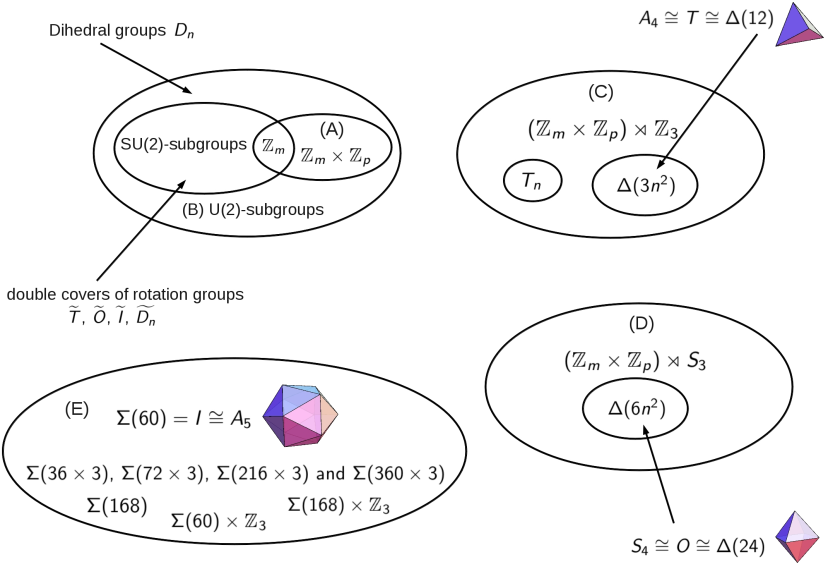
$\Rightarrow$ Constraints on observables
$\rightarrow$ testable in experiment.

Three-dimensional unitary representations $\rightarrow$ if faithful representation: Flavour symmetry group is a subgroup of U(3).

$\rightarrow$ Want to know all possibilities for flavour symmetries in this framework.

- Finite subgroups of U(3) not classified.
- But: finite subgroups of SU(3) by now all classified.

The finite subgroups of SU(3)



Summary

- Mathematics is the language of physics $\rightarrow$ Symmetries of nature $\leftrightarrow$ symmetry groups.
- Many symmetries are realized exactly (or approximately) in nature!
- Many areas where group theory is applied in theoretical physics.
- Knowing the symmetries of problems allows to understand structure and classification of solutions.
- Mostly relevant in physics: representation theory of groups.
- Commonly used computer algebra system: **GAP** (Groups, Algorithms and Programming).
- SmallGroups library: groups up to order 2000 $\rightarrow$ group scans possible.
- Example: Flavour symmetries: Group theory needed to construct models and to classify possible flavour symmetry groups.

Thank you for your attention!