The Physics of String Instruments

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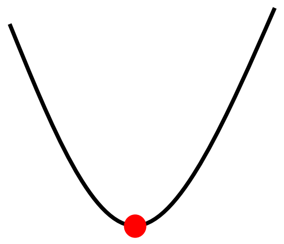
Vibration

- Object wants to be in a state of minimal potential energy.
- The force acting on it is equal to the negative gradient of its potential energy:

$$F = -\frac{dV}{dx}$$

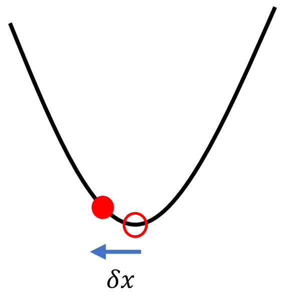
- Object at rest in position of minimal potential energy at x_0 $\rightarrow$

$$F = -\frac{dV}{dx}\bigg|_{x_0} = 0$$



Vibration

- Object moved away from its equilibrium position by a small amount δx



- $\underbrace{F(x_0 + \delta x)}_{m\frac{d^2\delta x}{dt^2}} = -\underbrace{\frac{dV}{dx}\Big|_{x_0}}_{0} - \underbrace{\frac{d^2V}{dx^2}\Big|_{x_0}}_{\equiv k}\delta x + O(\delta x^2) \approx -k\ \delta x$

$$m\frac{d^2x}{dt^2} = -k\ x$$

Harmonic oscillator

Vibration

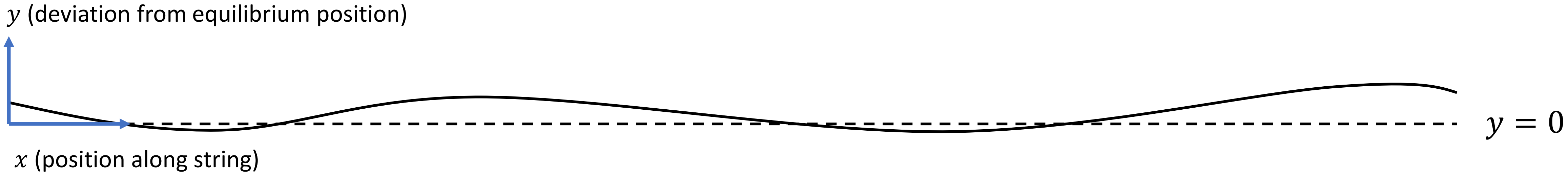
$$m\frac{d^2x}{dt^2} = -k\ x$$

Harmonic oscillator

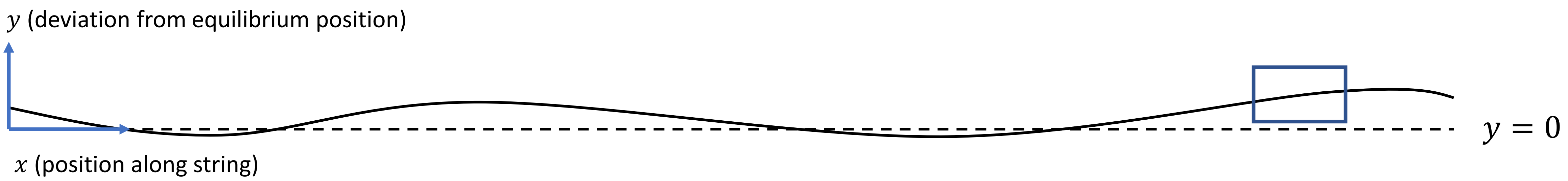
- **Solutions: Periodic functions in time**
- $x(t) = x_0 \cos(\omega t) + \frac{\dot{x}_0}{\omega}\sin(\omega t)$ with $\omega = \sqrt{k/m}$

Vibrating string

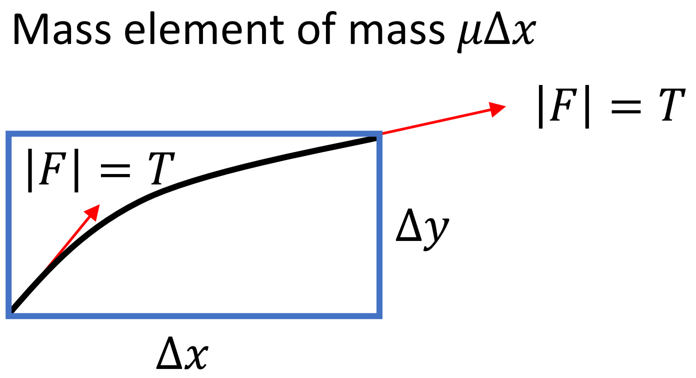
- Oscillating object: 1-dimensional (one degree of freedom)
- Vibrating string: Infinite number of degrees of freedom $y(x)$



Vibrating string



- String is under a tension (force) T constant along the whole string
- Mass per length of the string μ

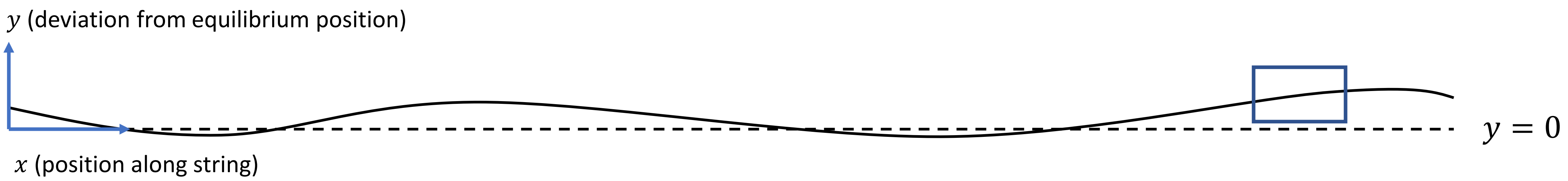


Force is tangential to string: $F(x) = T\frac{dy}{dx}$

Net force in y-direction: difference of the forces at the ends

$$T\left(\left.\frac{dy}{dx}\right|_{x+\Delta x} - \left.\frac{dy}{dx}\right|_{x}\right) = T\left.\frac{d^2y}{dx^2}\right|_{x}\Delta x + O(\Delta x^2)$$

Vibrating string



- String is under a tension (force) T constant along the whole string
- Mass per length of the string μ
- Net force in y-direction:

$$\underbrace{F_y(x)}_{m\frac{d^2y}{dt^2} = \mu\Delta x\,\frac{d^2y}{dt^2}} = T\,\frac{d^2y}{dx^2}\bigg|_x \Delta x$$

$$\frac{\partial^2 y}{\partial t^2} - \frac{\mu}{T}\frac{\partial^2 y}{\partial x^2} = 0$$

Wave equation

Vibrating string

- Described by the wave equation: Hyperbolic PDE
- String vibrations can mathematically be described as superpositions of waves moving along the string!

- Now we turn to musical instruments: String fixed at both ends.

Strings fixed at both ends

- Solve PDE by separation of variables $\dfrac{\partial^2 y}{\partial t^2} - \dfrac{\mu}{T}\dfrac{\partial^2 y}{\partial x^2} = 0$
- Ansatz: $y(x,t) = f(x)g(t)$
- $\frac{T}{\mu}\frac{1}{g(t)}\frac{d^2 g}{dt^2} = \frac{1}{f(x)}\frac{d^2 f}{dx^2}$
- $\frac{d^2 f}{dx^2} = Af$ with $f(0) = f(L) = 0$ Sturm-Liouville boundary value problem

(Eigenvalue problem → Discrete values for A)

- $\frac{d^2 g}{dt^2} = \frac{\mu}{T} Ag$

Strings fixed at both ends

- $\frac{d^2 f}{dx^2} = Af$ with $f(0) = f(L) = 0$ Sturm-Liouville boundary value problem

(Eigenvalue problem → Discrete values for A)

$f(x)$ is periodic in x.

- $\frac{d^2 g}{dt^2} = \frac{\mu}{T} Ag$ **$g(t)$ is periodic in t.**

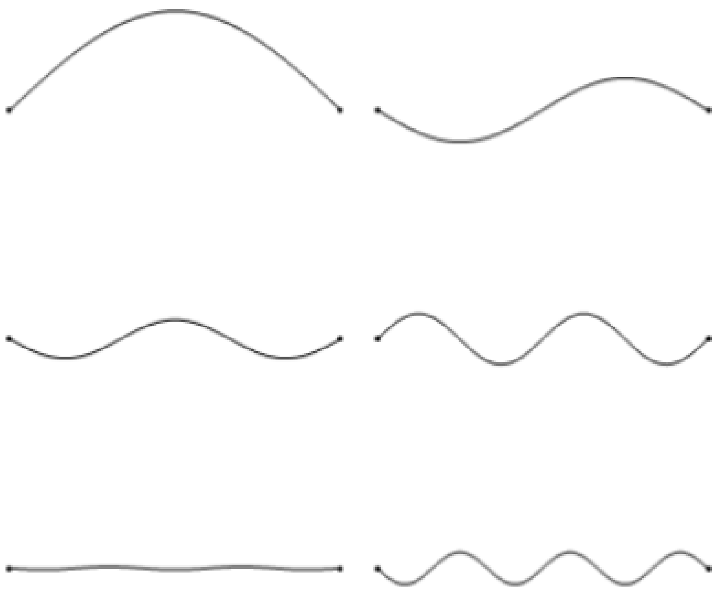
Compare to harmonic oscillator: $m\frac{d^2x}{dt^2} = -k\,x$

Standing waves

Amplitude at x

Oscillation in time (the same at all x)

- Strings fixed at both ends: $y(x,t) = f(x)g(t)$
- Periodic in space and time



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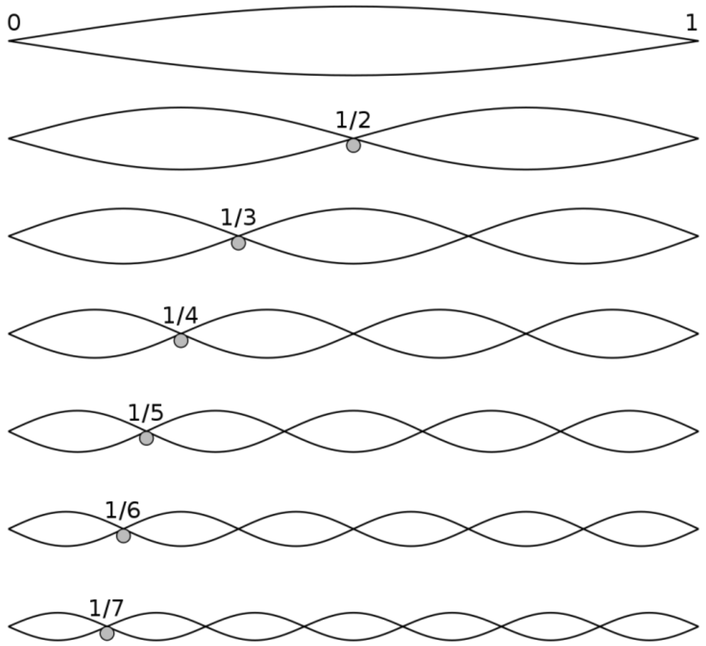
Standing waves

- $\frac{d^2 f}{dx^2} = Af$ with $f(0) = f(L) = 0$ Sturm-Liouville boundary value problem
- Solutions for f for different eigenvalues A are the modes of vibration

- Zeroth mode has largest wavelength $\lambda = 2L$
- 1st mode has $\lambda = L = 2L/2$
- 2nd mode has $\lambda = \frac{2L}{3}$
- nth mode has $\lambda = 2L/(n + 1)$



Frequencies of these harmonics

- All waves on the string have the same velocity of propagation c

$$\frac{\partial^2 y}{\partial t^2} - \underbrace{\boxed{\frac{\mu}{T}}}_{\frac{1}{c^2}} \frac{\partial^2 y}{\partial x^2} = 0$$

μ mass per length of string
T string tension
λ wavelength (=$2L/(n+1)$)
ν frequency

- Velocity $c = \sqrt{\frac{T}{\mu}} = \lambda\nu$
- Frequency $\nu = \frac{1+n}{2L}\sqrt{\frac{T}{\mu}}$

Higher frequencies for

- Higher string tension
- Lower string length
- Higher mass of the string
- Higher mode (overtones)

Typical values (guitar)

- String length 650 mm

- Example: E-String (lowest string of the guitar)
  - $\nu \approx 81\ \mathrm{Hz}$
  - $L = 650\ mm \Rightarrow \lambda = 1.3\ m$
  - $c = \lambda\nu \approx 105\ m/s$
  - $\mu \approx 5\ \mathrm{g/m} = 0.005\ \mathrm{kg/m}$
  - $T = \mu c^2 \approx 55\ N$

Experiments

Higher frequencies for

- Higher string tension
- Lower string length
- Higher mass/length of the string
- Higher mode (overtones)

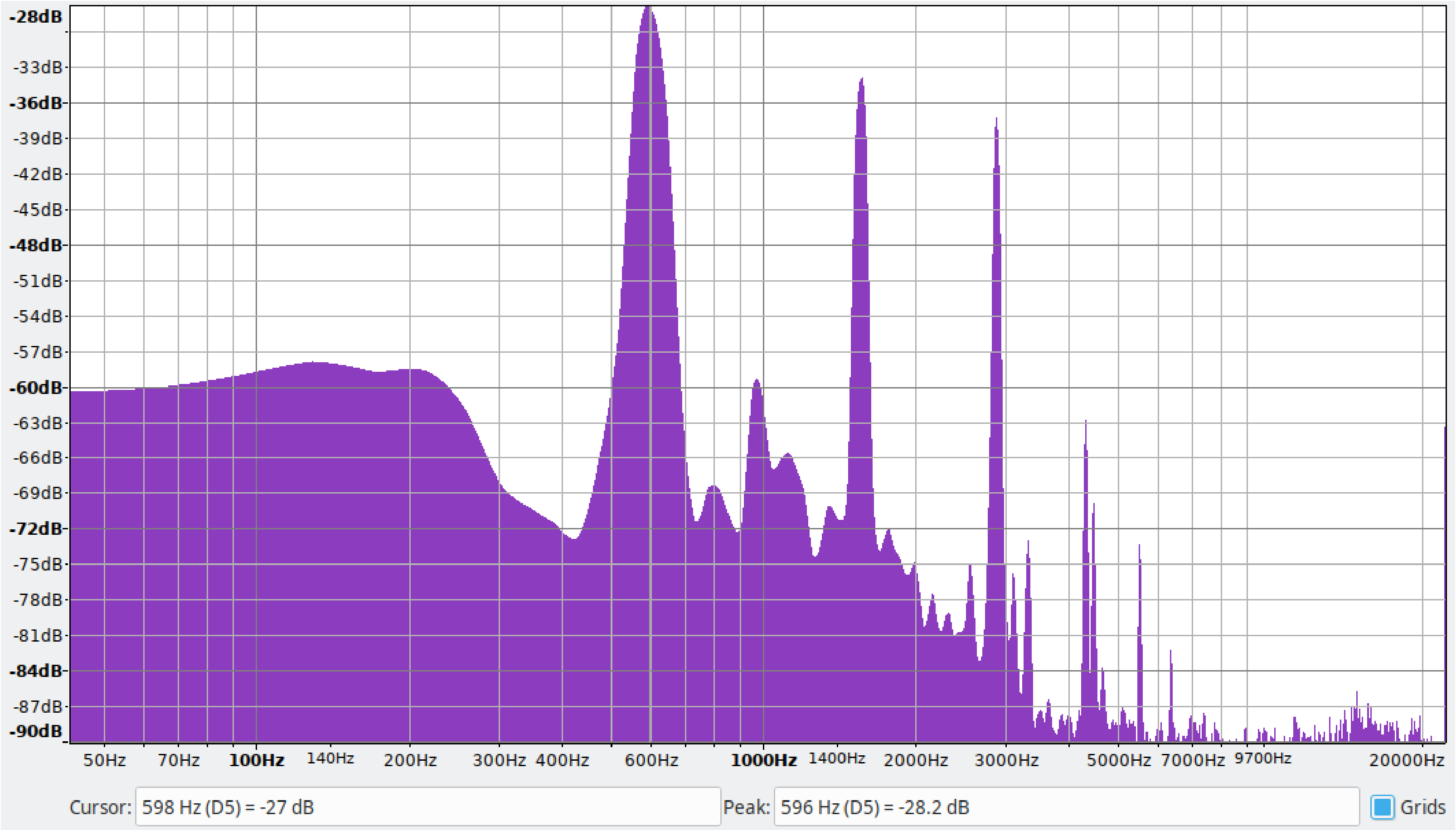
What makes the sound of an instrument?

- Plucking a string in practice always produces also higher modes
- Distribution of these modes makes the character of the sound
- **Player can control this!**

Examples for overtone spectra

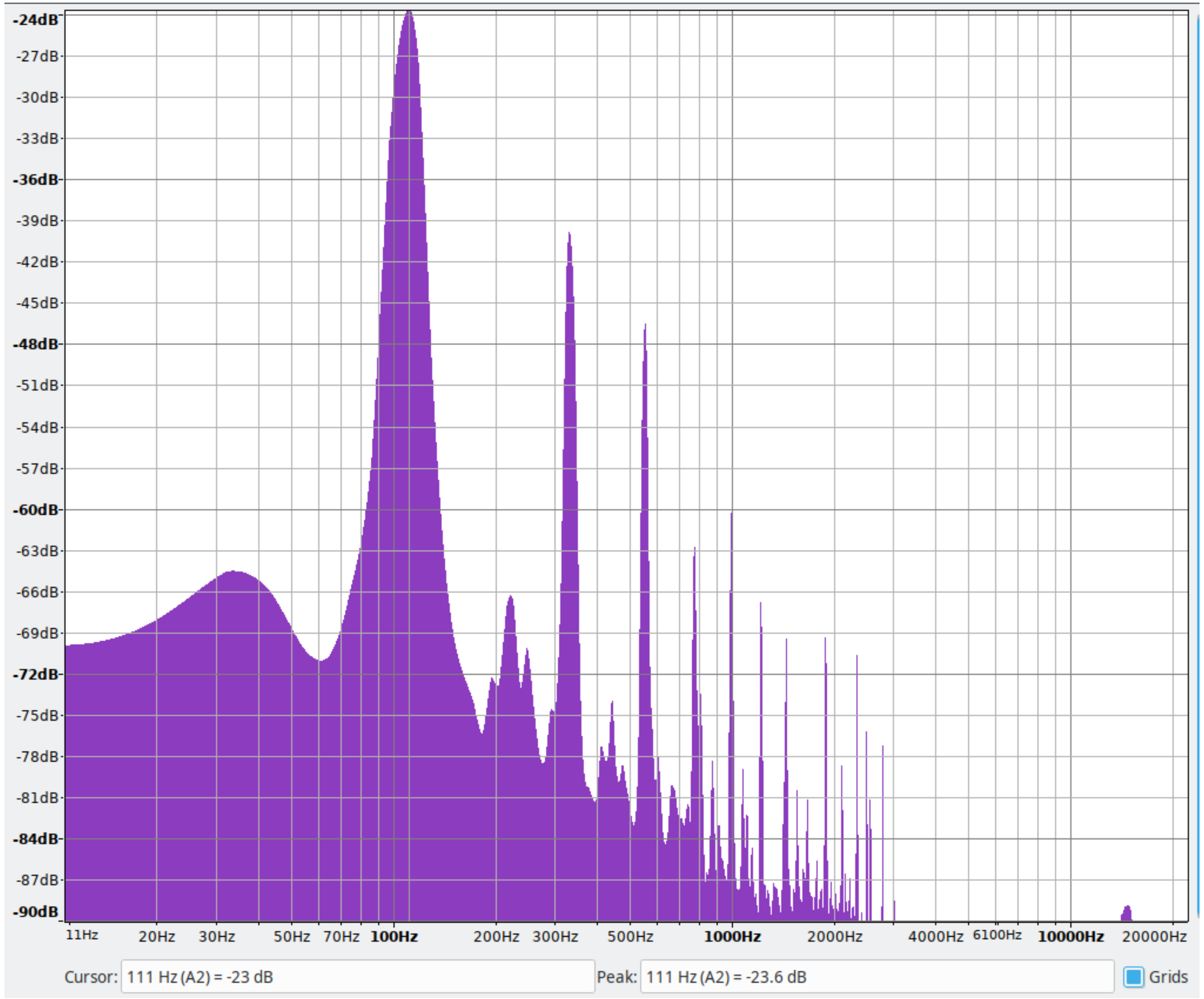
- Wine glass (quite strong zeroth mode)

596 Hz



Examples for overtone spectra

- A string of guitar (base frequency 110 Hz)
- Plucked at center of string



Examples for overtone spectra

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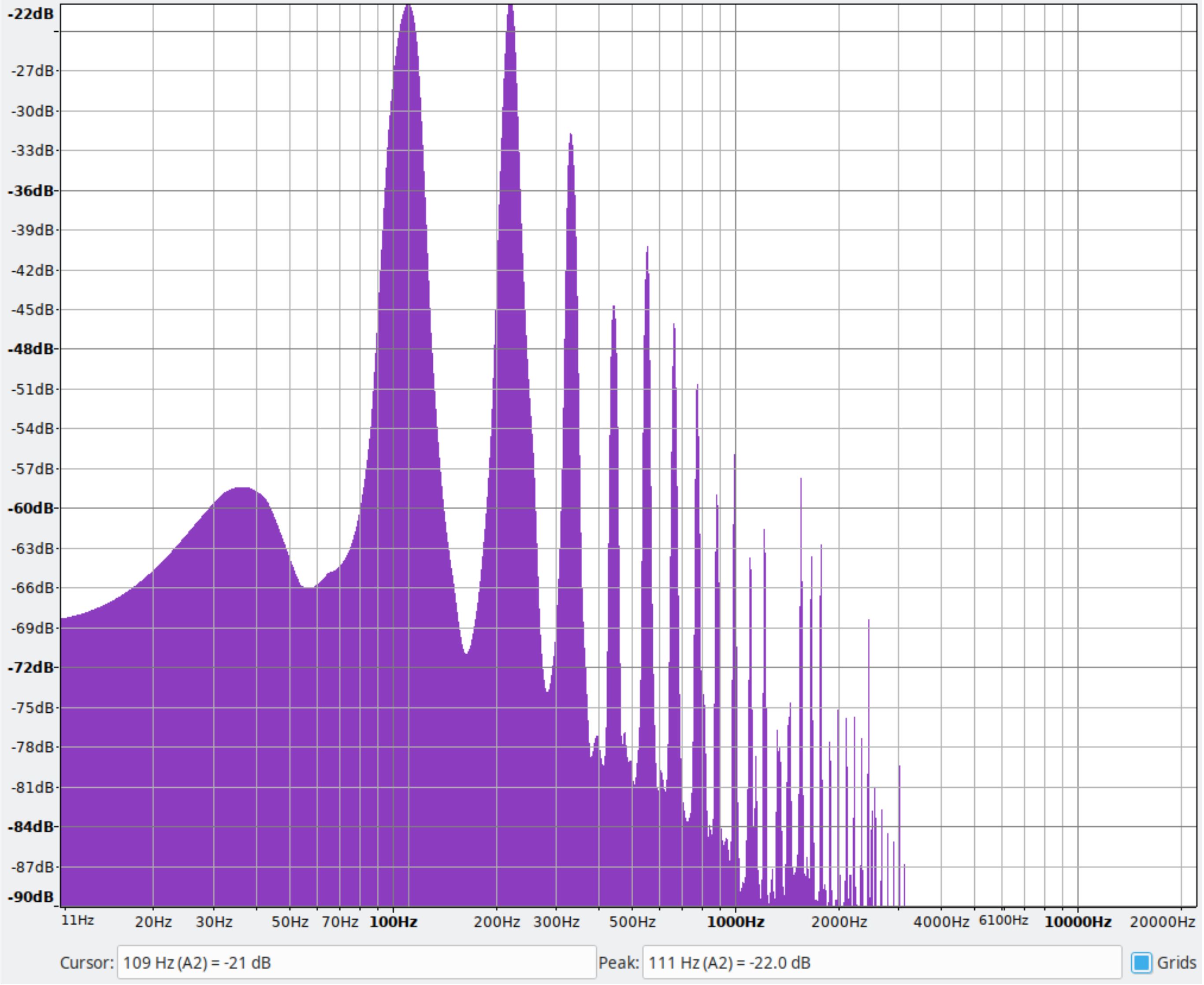
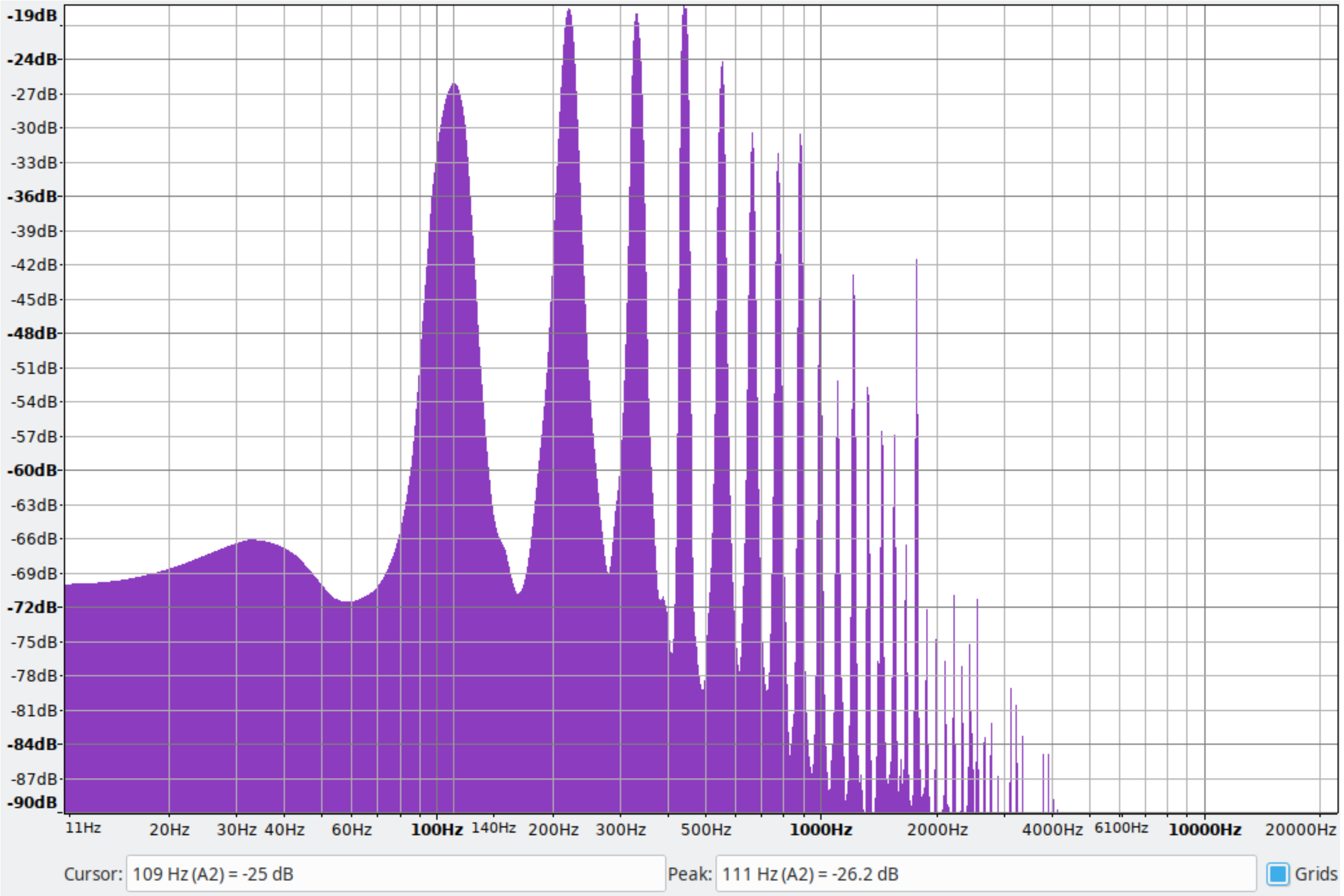


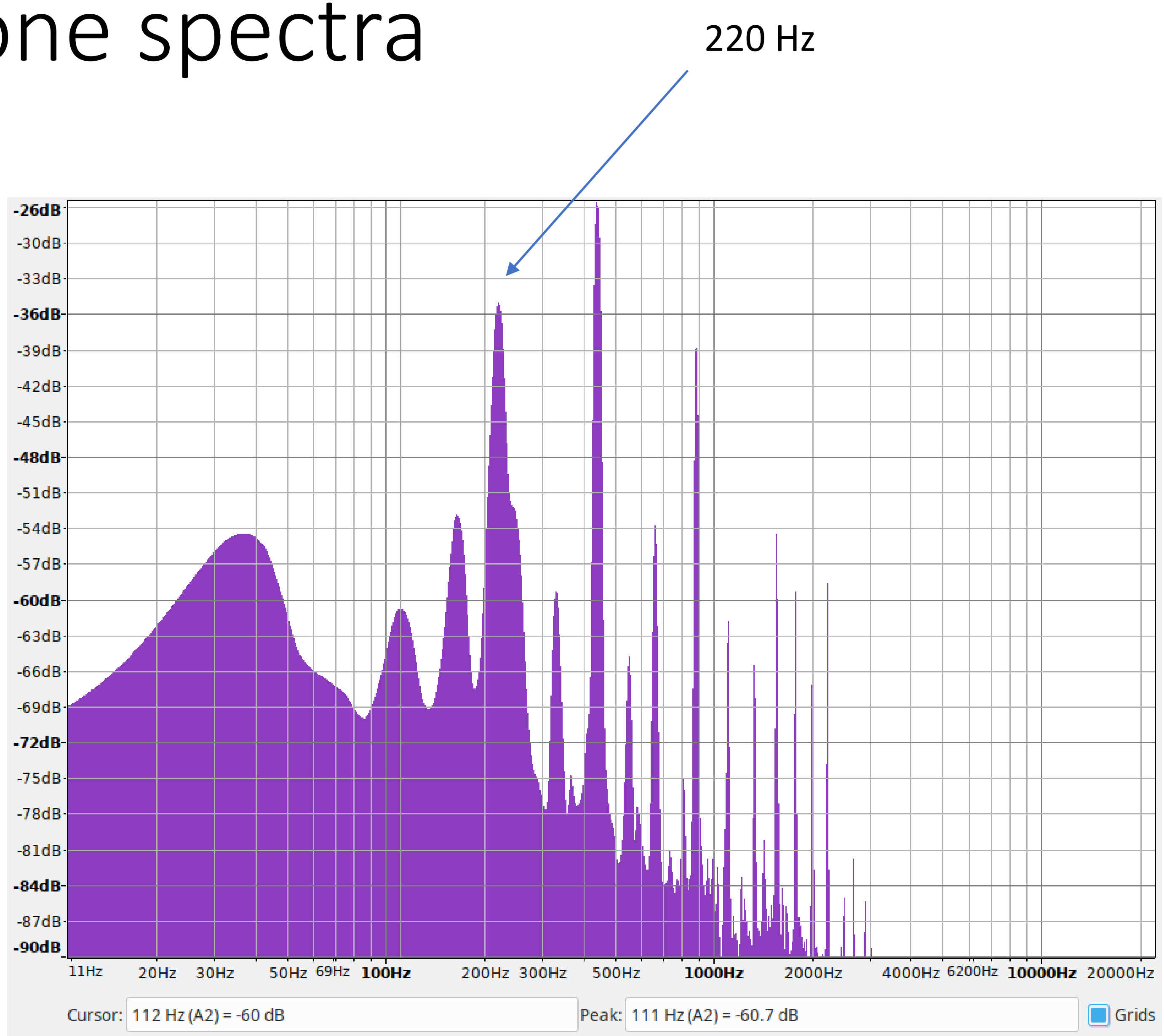
Examples for overtone spectra

- A string of guitar (base frequency 110 Hz)
- Plucked close to edge of string



Examples for overtone spectra

- A string of guitar
  (base frequency 110 Hz)
- Harmonic (Flageolet)



What makes the sound of an instrument?

- Way how the string is hit determines the oscillation with which the body of the instrument is “fed”
- Sound we hear is (almost) **not** the sound of the strings, but the sound of the vibrating body of the instrument, set into vibration by the strings
- Vibration of the body: Same mathematical structure as the vibrating string, but in 2d → Standing waves in two dimensions